## Lecture Ch. 4b

- Hydrostatic equilibrium
  - Special cases
  - Pressure altitude dependence
- More Midterm Review problems
  - Terminology review

Curry and Webster, Ch. 4 (pp. 96-115; skip 4.5 (except 4.5.1), 4.6)
Tuesday, Oct. 20: Homework, Review (bring questions), and Read Ch. 5
Tuesday, Oct. 27: Midterm
Thursday, Oct. 29: meet to work on ROAST!

## Water Vapor Metrics

The *water vapor mixing ratio*, $w_v$, is the ratio of the mass of water vapor present to the mass of dry air. It is thus defined, after substituting from the ideal gas law, as

$$w_v = \frac{m_v}{m_d} = \frac{\rho_v}{\rho_d} = \varepsilon\frac{e}{p-e} \quad (4.36)$$

where $\varepsilon = M_v/M_d = 0.622$ (Section 1.7). A value of the *saturation mixing ratio*, $w_s$, is given by

$$w_s = \varepsilon\frac{e_s}{p-e_s} \quad (4.37)$$

Since $p >> e$ and $p >> e_s$,

$$\mathcal{H} \approx \frac{w_v}{w_s} \quad (4.38)$$

is an approximate definition of the relative humidity.

The water vapor mixing ratio can be related to the specific humidity, $q_v$, which was originally defined in Section 1.7, as

$$q_v = \frac{m_v}{m_d+m_v} = \varepsilon\frac{e}{p-(1-\varepsilon)e} = \frac{w_v}{1+w_v} \quad (4.39)$$

Since both $w_v$ and $q_v$ are always smaller than 0.04, $q_v \approx w_v$.

## Special Cases of Hydrostatic Equilibrium

- 1. rho=constant (homogeneous)
  - H=8 km =RT/g=scale height eq. 1.39
- 2. constant lapse rate (implied if hydrostatic, homogeneous, and ideal gas)
  - -dT/dz=constant=-g/R=-34/deg/km
- 3. isothermal T=constant (and ideal gas)
  - p=p_0*exp(-z/H)

## Special Cases of Hydrostatic Equilibrium

- Hydrostatic: Force balance on gravity and upward pressure

tion of decreasing pressure (upwards). The vertical pressure gradient force is generally in very close balance with the downward force due to gravitational attraction. This is called *hydrostatic balance*, and is written as

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad (1.33)$$

where $g$ is the acceleration due to the Earth's gravity. The hydrostatic balance is applicable to most situations in the atmosphere and ocean, exceptions arising in the presence of large vertical accelerations such as are associated with thunderstorms.

## Homogeneous Atmosphere

**Homogeneous**
**Constant density**
**Constant lapse rate**

Because air is compressible and density decreases with height in the atmosphere (Figure 1.5), integration of (1.34) for the atmosphere is more complicated than for the ocean. However, useful insights can be derived from examining an idealized *homogeneous atmosphere*, where density is assumed constant. Consideration of a homogeneous atmosphere with finite surface pressure implies a finite total height for the atmosphere, which is called the *scale height H*. Assuming that density is constant, we can integrate (1.34) from sea level, where the pressure is $p_0$, to a height $H$, where the pressure is zero, to obtain

$$p_0 = \rho g H \quad (1.38)$$

The height of the homogeneous atmosphere (often referred to as the *scale height*) is therefore

$$H = \frac{p_0}{\rho g} = \frac{R_d T_0}{g} \quad (1.39)$$

where $T_0$ is the surface temperature and $H$ can be evaluated from the surface temperature and known constants to be approximately 8 km. From the ideal gas law, it is easily inferred that temperature must decrease with height in the homogeneous atmosphere. The lapse rate of the homogeneous atmosphere is obtained by differentiating the ideal gas law with respect to $z$, holding density constant

$$\frac{\partial p}{\partial z} = \rho R_d \frac{\partial T}{\partial z}$$

Combining (1.40) with the hydrostatic equation (1.33) leads to the resul

$$\Gamma = -\frac{\partial T}{\partial z} = \frac{g}{R_d} = 34.1°\text{C km}^{-1}$$

The lapse rate for the homogeneous atmosphere is referred to as the *autoconvective lapse rate* for the following reason: if the lapse rate exceeds the autoconvective value, it is implied that the lower air is less dense than the air above, causing the atmosphere to overturn and the spontaneous initiation of convection. Values of the atmospheric lapse rate as large as the autoconvective value are observed over desert surfaces in summer when the solar heating is high; however, lapse rates in the atmosphere typically do not exceed $\Gamma \approx 10°\text{C km}^{-1}$.

## Isothermal Atmosphere

Further insight is gained by examining the characteristics of yet another idealized atmosphere, called the *isothermal atmosphere*. After substitution of the ideal gas law for density, we can write the hydrostatic equation in the following form:

$$\partial p = -\frac{pg}{R_d T}\partial z \quad (1.42)$$

This equation is easily integrated for a constant temperature from sea level ($z = 0$, $p = p_0$) to some arbitrary height $z$

$$\int_{p_0}^{p}\frac{dp}{p} = -\frac{g}{R_d T}\int_0^z dz \quad (1.43)$$

or

$$\ln\frac{p}{p_0} = -\frac{gz}{R_d T} \quad (1.44a)$$

Taking antilogs and using $H = RT/g$, we have

$$p = p_0 \exp\left(-z/H\right)$$

Thus pressure decreases exponentially with height in an isothermal atmosphere, and there is no definite upper boundary to this atmosphere. Note that when $z = H$, the pressure is $1/e$ of its surface value. The isothermal atmosphere resembles the real atmosphere more closely than does the homogeneous atmosphere; however, (1.44b) is not applicable to the real atmosphere except when applied over a shallow layer above the ground.

# Hydrostatic Equilibrium Example

Consider a planet with an atmosphere in hydrostatic equilibrium. Assume that the atmosphere is an ideal gas. Also assume that the temperature is a maximum at the surface of the planet, and, as height increases, the temperature in the atmosphere decreases linearly (in other words, temperature decreases with height at a constant rate). Derive a formula for atmospheric density as a function of height in this atmosphere.

# Quiz

Answer briefly and clearly, with appropriate equations or diagrams.

- If a homogeneous mixture of two substances coexists in liquid and vapor phases at a series of pressures (each of which corresponds to exactly one temperature), how many degrees of freedom are in this system?
- What is thermal equilibrium? Give an equation.
- Give the equation for the Gibbs phase rule.
- What is the change in free energy for a phase change?
- What type of pressure change is described by the Clausius-Clapeyron equation? i.e. what changes as a function of what under what conditions?

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# Answers

- If a homogeneous mixture of two substances coexists in liquid and vapor phases at a series of pressures (each of which corresponds to exactly one temperature), how many degrees of freedom are in this system?
  - 1
- What is thermal equilibrium? Give an equation.
  - $T_1=T_2$
- Give the equation for the Gibbs phase rule.
  - f=c+2-p
- What is the change in free energy for a phase change?
  - 0 (at constant T and P)
- What type of pressure change is described by the Clausius-Clapeyron equation? i.e. what changes as a function of what under what conditions?
  - $p_s=f(T)$ for l/v saturation pressure

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# Hydrostatic Equilibrium Example

Consider a planet with an atmosphere in hydrostatic equilibrium. Assume that the atmosphere is an ideal gas. Also assume that the temperature is a maximum at the surface of the planet, and, as height increases, the temperature in the atmosphere decreases linearly (in other words, temperature decreases with height at a constant rate). Derive a formula for atmospheric density as a function of height in this atmosphere.

From the hydrostatic equation for an ideal gas (Eqn. 1.42)

$$\partial p = -\frac{pg}{R_d T}\partial z$$

and a constant lapse rate $\Gamma = -\frac{dT}{dz}$ we get

$$dp = -\frac{pg}{R_d T}dz\left(\frac{-dT/dz}{\Gamma}\right) = -\frac{pg}{\Gamma R_d}\frac{dT}{T}$$

$$\frac{dp}{p} = \left(\frac{g}{\Gamma R_d}\right)\frac{dT}{T}$$

$$\int\frac{dp}{p} = \int\left(\frac{g}{\Gamma R_d}\right)\frac{dT}{T}$$

$$\ln\frac{p}{p_0} = \left(\frac{g}{\Gamma R_d}\right)\ln\frac{T}{T_0}$$

$$p = p_0\left(\frac{T}{T_0}\right)^{\left(g/R_d\Gamma\right)}$$

which is Eqn 1.48. Then dividing both sides by RT and noting that for an ideal gas

$\rho = \frac{p}{RT}$, we get

$$\frac{p}{RT} = \rho = \frac{p_0}{RT}\left(\frac{T}{T_0}\right)^{g/R_d\Gamma} = \frac{p_0}{R(T_0-\Gamma z)}\left(\frac{T_0-\Gamma z}{T_0}\right)^{g/R_d\Gamma}$$

# Clausius Clapeyron Example

The saturation vapor pressure at a temperature of 30°C is 42.4 hPa. The gas constant for dry air is 287 J $K^{-1}kg^{-1}$. The gas constant for water vapor is 461 J $K^{-1}kg^{-1}$.

In addition to the constants given above, here is one more: the saturation vapor pressure at a temperature of 40°C is 73.8 hPa. Assuming that the latent heat of vaporization is constant, use this information to calculate the numerical value for this latent heat.

The Clausius Clapeyron equation can be integrated if L is assumed constant, and the result is Eqn. 4.23. Using 30°C=303K and 40°C=313K, and knowing saturation vapor pressure values for each, the only unknown is L. Solving Eqn. 4.23,

$$e_2 = e_1\exp\left[-\frac{L_{lv}}{R_v}\left(\frac{1}{T_2}-\frac{1}{T_1}\right)\right]$$

$$L_{lv} = -R_v\left(\frac{T_1T_2}{T_1-T_2}\right)\left(\ln\frac{e_2}{e_1}\right) = -461\left(\frac{303\times 313}{303-313}\right)\left(\ln\frac{73.8}{42.4}\right) = 2.4\times 10^6$$

$L=2.4x10^6$ J/kg.

# Degrees of Freedom Example

Name the five main components of the atmosphere. (a) If all components are in the gas phase, how many degrees of freedom are there in the system? (b) If water condenses or freezes, does that number increase or decrease? (c) If new components are added by pollution, how does that change (i) the number of possible phases and (ii) the degrees of freedom of the atmosphere?

The five main components of the atmosphere are nitrogen ($N_2$), oxygen ($O_2$), carbon dioxide ($CO_2$), argon (Ar), and water ($H_2O$).

(a) For this system, we can use the Gibbs phase rule (Eqn. 4.2) with χ=5, φ=1, f=χ-φ+2=6.

(b) Condenses φ=2, f=5 [decrease]; freezes φ=2, f=5 [decrease] (both: φ=3, f=4 [decrease]).

(c) (i) number of phases that can exist at atmospheric pressure may increase with additional components, since multiple liquid and solid phases may form; (ii) degrees of freedom increase with the number of components and will decrease with the number of phases.

# Quiz

Answer briefly and clearly, with appropriate equations or diagrams.

- If a pure substance coexists in liquid and vapor phases at a series of pressures (each of which corresponds to exactly one temperature), how many degrees of freedom are in this system?
- What is the relationship between two pressures at mechanical equilibrium?
- Give the equation for the Gibbs phase rule.
- What three types of equilibrium are required by the Gibbs phase rule?