Simple Mathematical Models of the Greenhouse Effect, and Global Warming

Mathematical Models

- Scientists often use mathematical and computer models to understand complex systems (like Earth's climate)
- A model uses equations to represent essential aspects of the system
- The equations describe how one part of the system is quantitatively related to another

Mathematical Models

- Models are necessarily oversimplified and cannot accurately and comprehensively represent all details of a complex system
- Because models are simple, it is easier to see how the real system approximately works
- More equations can be added to a model until it becomes too difficult to understand or simulate on a computer

The Simplest Model

- The Earth absorbs solar radiation
  $S_0 / 4$

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- The Earth has a certain planetary albedo for solar wavelengths

  $(1 - \alpha_p) S_0 / 4$

The Simplest Model

- The Earth absorbs solar radiation

  $S_0 / 4$
- The Earth has a certain planetary albedo for solar wavelengths

  $(1 - \alpha_p)\, S_0 / 4$
- The Earth emits like a blackbody at IR wavelengths

  $\sigma\, T_e^4$

  $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}

The Simplest Model

- The Earth has no atmosphere

  T_e *is the surface temperature*

The Simplest Model

- The Earth has no atmosphere

  T_e is the surface temperature
- The Earth is in radiative balance

  $(1 - \alpha_p)\ S_0 / 4 = \sigma\ T_e^4$

The Simplest Model

We now have our simple model of Earth's climate:

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_e^4$$

What is the value of T_e?

Rearrange the equation so that T_e is on the left side and everything else is on the right side

What is the value of T_e?

Rearrange the equation so that T_e is on the left side and everything else is on the right side

$$T_e = \left[\frac{(1 - \alpha_p) S_0}{4\sigma} \right]^{1/4}$$

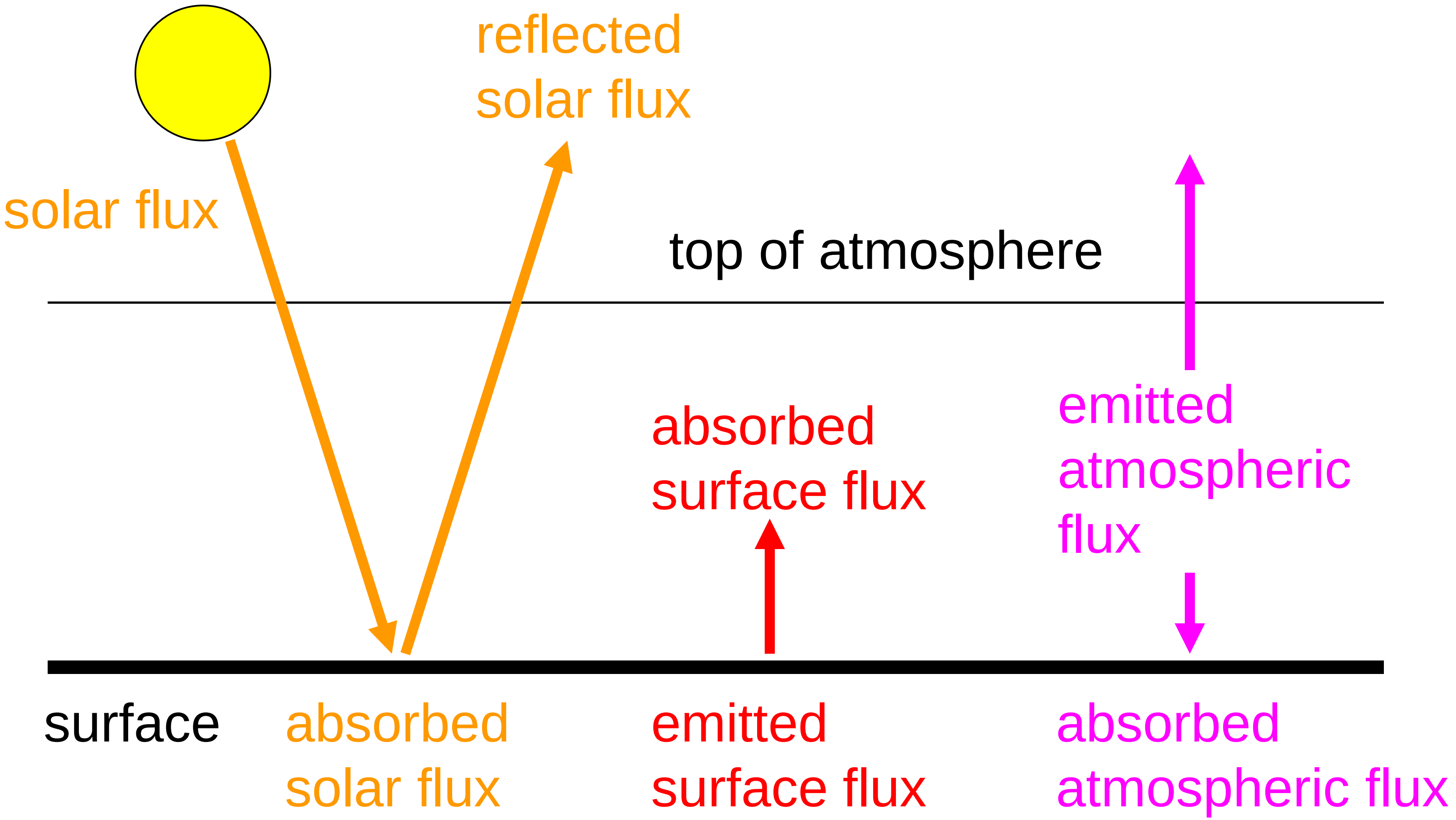
Excel spreadsheet

http://meteora.ucsd.edu/~jnorris/cosmos/cosmos.html

A Simple Atmosphere

- Now add an atmosphere to the model
- This simple atmosphere is perfectly transmissive at solar wavelengths and perfectly absorptive at IR wavelengths
- The atmosphere emits as a perfect blackbody at IR wavelengths
- The atmosphere and surface can have different temperatures T_a and T_s

A Simple Atmosphere



Top Radiative Balance

- No IR radiation emitted by the surface is transmitted to space because all is absorbed by the atmosphere
- Downward radiation flux is $S_0 / 4$
- One component of upward flux is reflected solar radiation: $\alpha_p S_0 / 4$
- The other component of upward flux is IR radiation emitted by the atmosphere: σT_a^4

Top Radiative Balance

For radiative balance:

$$S_0 / 4 = \alpha_p S_0 / 4 + \sigma T_a^4$$

This can be rearranged as:

$$(1 - \alpha_p) S_0 / 4 = \sigma T_a^4$$

Note that T_a is the same as T_e

Atmosphere Radiative Balance

- The atmosphere absorbs no solar radiation
- The atmosphere absorbs all IR radiation emitted by the surface: σT_s^4
- The atmosphere emits IR radiation in both the upward and downward directions: $\sigma T_a^4 + \sigma T_a^4$

Atmosphere Radiative Balance

For radiative balance:

$\sigma\ T_s^4 = \sigma\ T_a^4$ (up) $+ \sigma\ T_a^4$ (down)

This can be rearranged as:

$\sigma\ T_s^4 = 2\ \sigma\ T_a^4$

Surface Radiative Balance

- The surface absorbs some solar radiation: $(1 - \alpha_p)\ S_0 / 4$
- The surface absorbs all IR radiation that the atmosphere emits downward: $\sigma\ T_a^4$
- The surface emits radiation: $\sigma\ T_s^4$

Surface Radiative Balance

For radiative balance:

$$(1 - \alpha_p)\, S_0 / 4 + \sigma\, T_a^4 = \sigma\, T_s^4$$

This can be rearranged as:

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_s^4 - \sigma\, T_a^4$$

The Simple Atmosphere

We now have a simple model of Earth's climate that includes an atmosphere:

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_a^4$$

$$\sigma\, T_s^4 = 2\, \sigma\, T_a^4$$

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_s^4 - \sigma\, T_a^4$$

What are values of T_a and T_s?

Obtain one equation such that T_a is on the left side, T_s is eliminated, and everything else is on the right side

Obtain a second equation such that T_s is on the left side, S_0 and α_p are eliminated, and everything else is on the right side

What are values of T_a and T_s?

$$T_a = \left[\frac{\left(1 - \alpha_p\right) S_0}{4\sigma} \right]^{1/4}$$

$$T_s = \left(2\right)^{1/4} T_a$$

What are values of T_a and T_s?

- Note that T_s is greater than T_a and T_e
- The atmosphere keeps the surface warmer than it would be if no atmosphere were present
- This is a mathematical model of the greenhouse effect

Excel spreadsheet

A Simple IR Window

- Now allow the atmosphere to transmit to space some of the IR radiation emitted by the surface
- Since the atmosphere is no longer perfectly absorptive at IR wavelengths, it also no longer emits as a perfect blackbody at IR wavelengths

A Simple IR Window

- Let ε be the fraction of IR radiation absorbed by the atmosphere

  surface radiation absorbed by atmosphere

  $\varepsilon \sigma T_s^4$

- Basic physics requires that the ε also be the fraction of IR radiation emitted by the atmosphere relative to a perfect blackbody

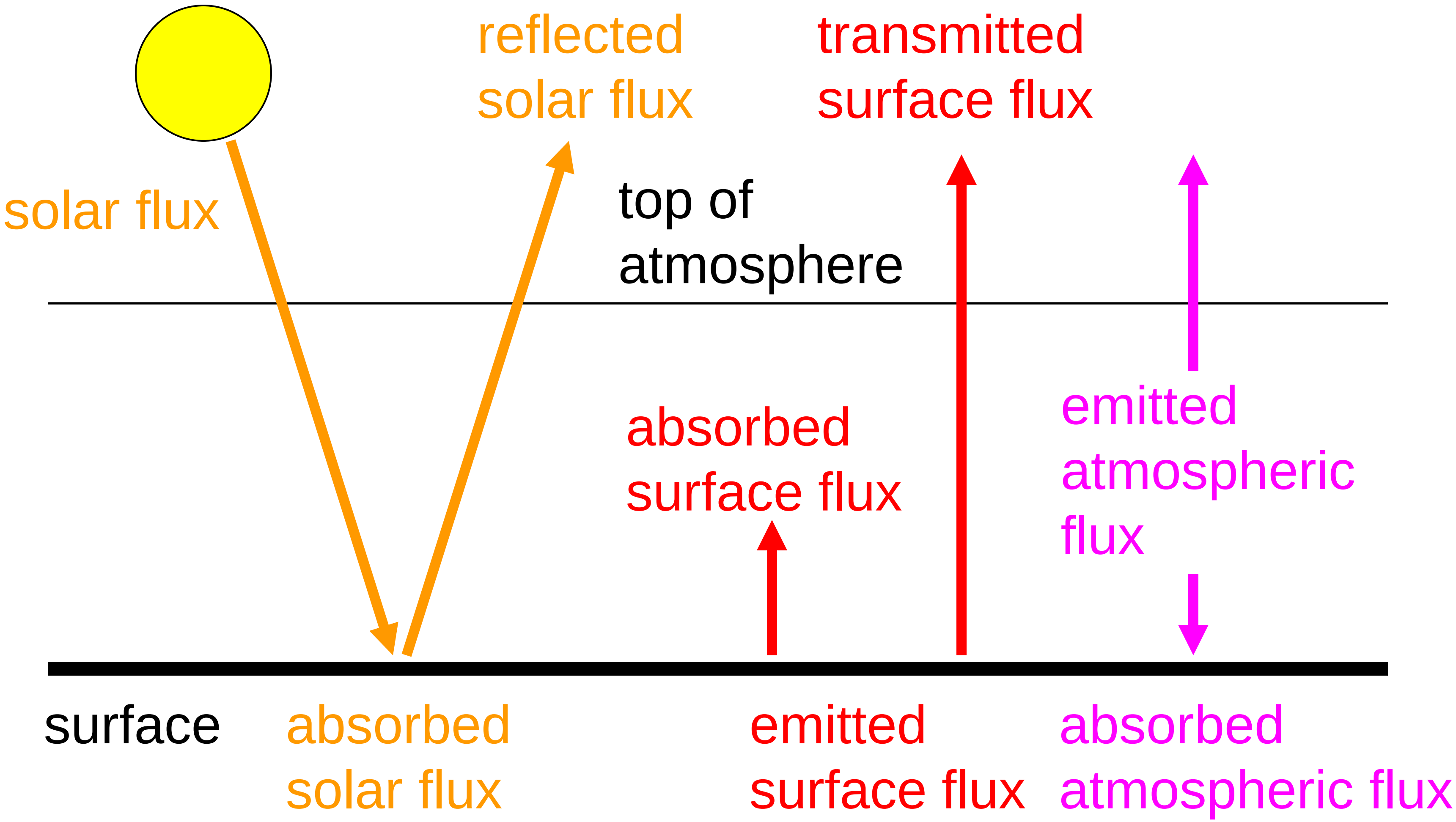
  radiation emitted upward and downward

  $\varepsilon \sigma T_a^4 + \varepsilon \sigma T_a^4$

Emissivity

- The parameter ε is called the emissivity of the atmosphere (the fraction of radiation emitted relative to a blackbody)
- The absorptivity of the atmosphere also has the value of ε (the fraction of incident radiation that is absorbed)

A Simple IR Window



Top Radiative Balance

- Downward radiation flux is S_0 / 4
- One component of upward flux is reflected solar radiation: $\alpha_p S_0 / 4$
- Another component is IR radiation emitted by the atmosphere: $\varepsilon \sigma T_a^4$
- A third component is IR radiation emitted by the surface and not absorbed by the atmosphere: $(1 - \varepsilon) \sigma T_s^4$

Top Radiative Balance

For radiative balance:

$S_0 / 4 = \alpha_p S_0 / 4 + \varepsilon \sigma T_a^4 + (1 - \varepsilon) \sigma T_s^4$

This can be rearranged as:

$(1 - \alpha_p) S_0 / 4 = \varepsilon \sigma T_a^4 + (1 - \varepsilon) \sigma T_s^4$

Note that T_a is no longer the same as T_e

Atmosphere Radiative Balance

- The atmosphere absorbs no solar radiation
- The atmosphere absorbs some IR radiation emitted by the surface: $\varepsilon \sigma T_s^4$
- The atmosphere emits IR radiation in both the upward and downward directions: $\varepsilon \sigma T_a^4 + \varepsilon \sigma T_a^4$

Atmosphere Radiative Balance

For radiative balance:

$\varepsilon\ \sigma\ T_s^4 = \varepsilon\ \sigma\ T_a^4$ (up) $+\ \varepsilon\ \sigma\ T_a^4$ (down)

This can be rearranged as:

$\varepsilon\ \sigma\ T_s^4 = 2\ \varepsilon\ \sigma\ T_a^4$

Surface Radiative Balance

- The surface absorbs some solar radiation: $(1 - \alpha_p)\ S_0 / 4$
- The surface absorbs all IR radiation that the atmosphere emits downward: $\varepsilon\ \sigma\ T_a^4$
- The surface emits radiation: $\sigma\ T_s^4$

Surface Radiative Balance

For radiative balance:

$$(1 - \alpha_p)\, S_0 / 4 + \varepsilon\, \sigma\, T_a^4 = \sigma\, T_s^4$$

This can be rearranged as:

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_s^4 - \varepsilon\, \sigma\, T_a^4$$

The Simple IR Window

We now have a simple model of Earth's climate that includes an atmosphere that is partially transmissive at IR wavelengths:

$$(1 - \alpha_p)\, S_0 / 4 = \varepsilon\, \sigma\, T_a^4 + (1 - \varepsilon)\, \sigma\, T_s^4$$

$$\varepsilon\, \sigma\, T_s^4 = 2\, \varepsilon\, \sigma\, T_a^4$$

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_s^4 - \varepsilon\, \sigma\, T_a^4$$

How are T_s and ε related?

Obtain one equation such that T_s is on the left side, T_a is eliminated, and everything else is on the right side

How are T_s and ε related?

Obtain one equation such that T_s is on the left side, T_a is eliminated, and everything else is on the right side

$$T_s = \left[\frac{(1-\alpha_p)S_0}{(4-2\varepsilon)\sigma}\right]^{1/4}$$

How are T_s and ε related?

- Note that $\varepsilon = 0$ corresponds to no effective atmosphere and $\varepsilon = 1$ corresponds to a perfectly absorbing atmosphere
- For $0 < \varepsilon < 1$, T_s is warmer than T_e and colder than T_s for the perfectly absorbing atmosphere
- Changes in ε represent changes in greenhouse gas concentrations

How are T_s and ε related?

- What value of ε will produce a value of T_s equal to current global surface temperature (about 288 K or 15°C)?
- How close is this to the real fraction of radiation absorbed by the atmosphere?
- How much does ε need to change to produce a 1 K increase in T_s?
- What about 2 K? 5 K? A decrease in T_s?

Excel spreadsheet

Non-Equilibrium

- The climate does not have an instantaneous response to a change in emissivity
- What is the transient behavior of the atmosphere before it comes to equilibrium?

Thermal Inertia

- The change in temperature caused by radiative imbalance does not occur instantaneously due to thermal inertia
- It takes decades to centuries for the ocean to warm up or cool down
- We have not yet experienced the full warming that will be produced by the CO_2 we have already put into the atmosphere

A Simple Climate Model

- Let the Earth be covered by a “swamp” ocean with uniform depth h, density ρ, and specific heat c

  thermal inertia of ocean = $\rho\, c\, h$

- Let F be the net radiation flux at the Earth’s surface

  $F = (1 - \alpha_p)\, S_0 / 4 + \varepsilon\, \sigma\, T_a^4 - \sigma\, T_s^4$

A Simple Climate Model

Let ΔT_s be the change in temperature across time interval Δt

$$\Delta T_s = F\,\Delta t\,/\,(\rho\, c\, h)$$

where

$$F = (1 - \alpha_p)\, S_0 / 4 + \varepsilon\, \sigma\, T_s^4 / 2 - \sigma\, T_s^4$$

A Simple Climate Model

- Assume ε is known as a function of time
- If T_s is known at time t_0, it is simple to calculate T_s at time $t_0 + \Delta t$
- The value of T_s at time $t_0 + \Delta t$ can then be used to calculate T_s at time $t_0 + 2\Delta t$
- Etc.

A Simple Climate Scenario

- Let the climate initially be in equilibrium
- Let ε change instantaneously from 0.8 to 0.85 and thereafter remain constant
- Let *h* be 4000 m (approximate average ocean depth)
- Let ρ be 1025 kg m^{-3} and *c* be 3850 J kg^{-1} K^{-1} (typical values for seawater)

Excel spreadsheet

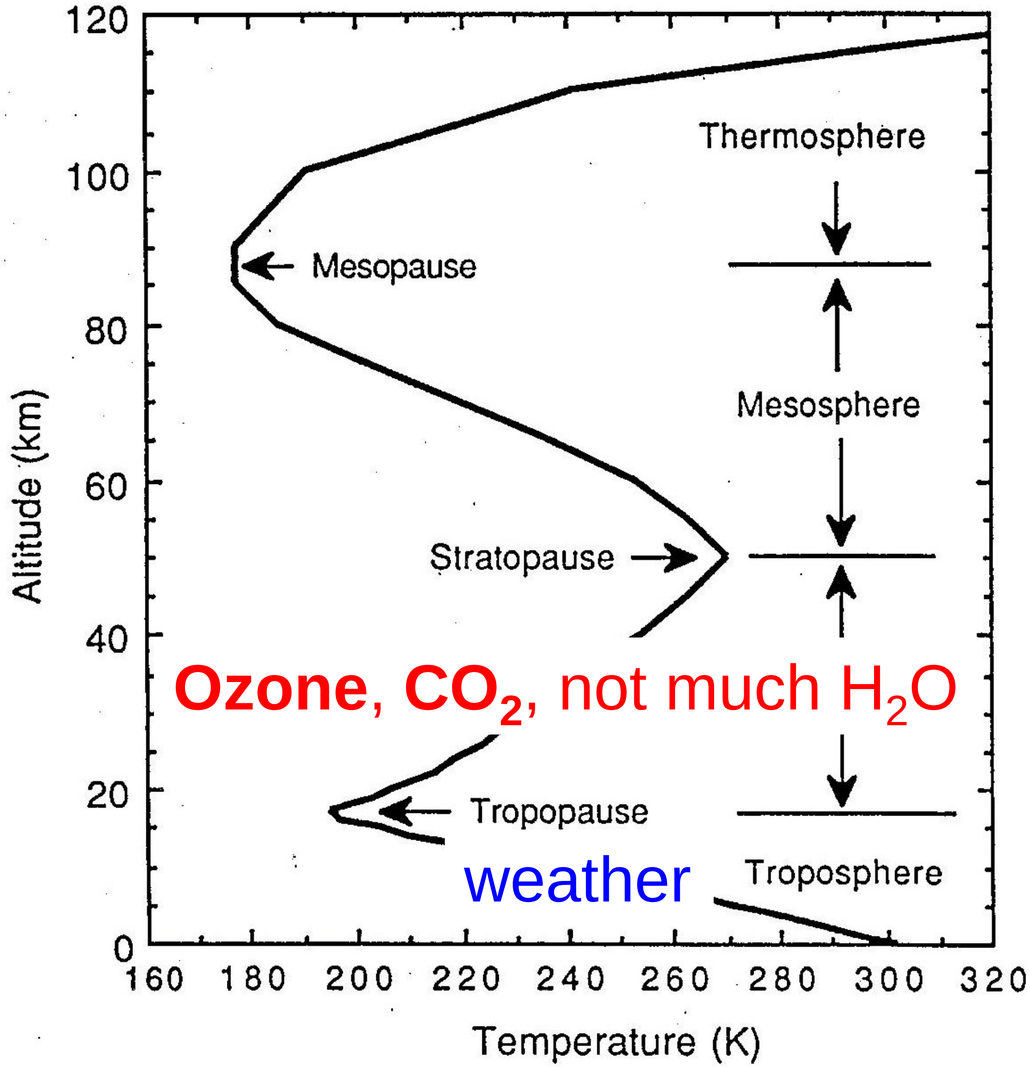
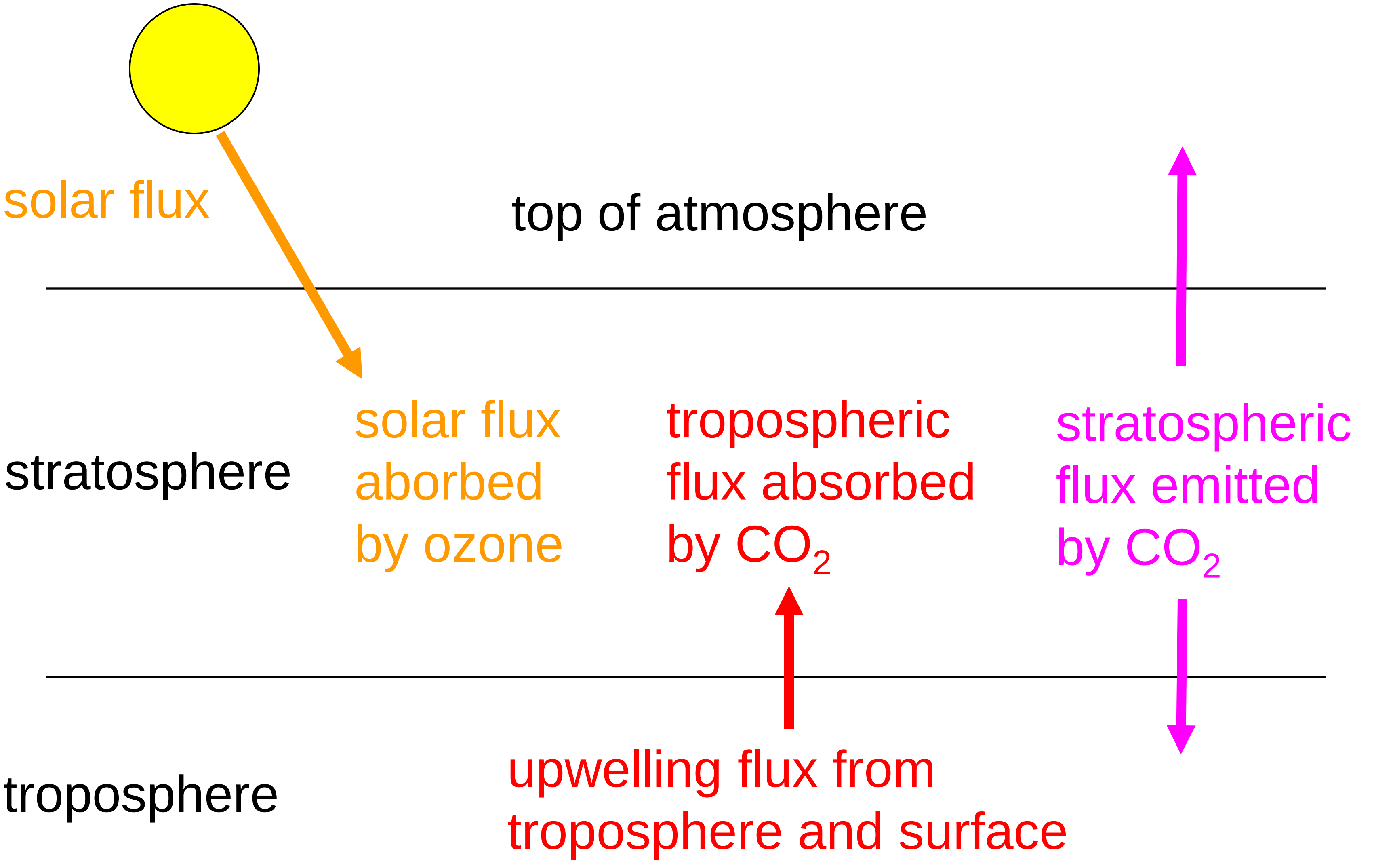


Fig. 1.2 The main zones of the atmosphere defined according to the temperature profile of the standard atmosphere profile at 15°N for annual-mean conditions. [Data from U.S. Standard Atmosphere Supplements (1966).]

From Hartmann

Radiative Balance of the Stratosphere



Radiative Balance of the Stratosphere

$\varepsilon_{O3}\, F_{\mathrm{solar}} + \varepsilon_{CO2}\, F_{\mathrm{trop}} = 2\, \varepsilon_{CO2}\, T_{strat}^{\;4}$

ε_{O3} is fraction absorbed by ozone

ε_{CO2} is fraction absorbed/emitted by CO_2

F_{solar} is downwelling solar flux

F_{trop} is upwelling thermal flux from troposphere

T_{strat} is stratospheric temperature

Divide into Five Teams

Radiative Balance of the Stratosphere

$$\varepsilon_{O3}\, F_{solar} + \varepsilon_{CO2}\, F_{trop} = 2\, \varepsilon_{CO2}\, T_{strat}^{4}$$

How will stratospheric temperature T_{strat} change if

- solar flux F_{solar} increases?
- ozone decreases?
- CO_2 increases?

How will surface temperature T_{sfc} change if

- solar flux increases?
- ozone decreases?
- CO_2 increases?

What Caused Recent Global Warming?

Natural

- Increase in solar flux

Not natural

- Increase in CO_2 by burning fossil fuels
- Decrease in ozone from CFCs

Theoretical Prediction

Solar flux increases

- Surface and troposphere warm/cool?
- Stratosphere warms/cools?

CO_2 increases

- Surface and troposphere warm/cool?
- Stratosphere warms/cools?

Ozone decreases

- Surface and troposphere warm/cool?
- Stratosphere warms/cools?

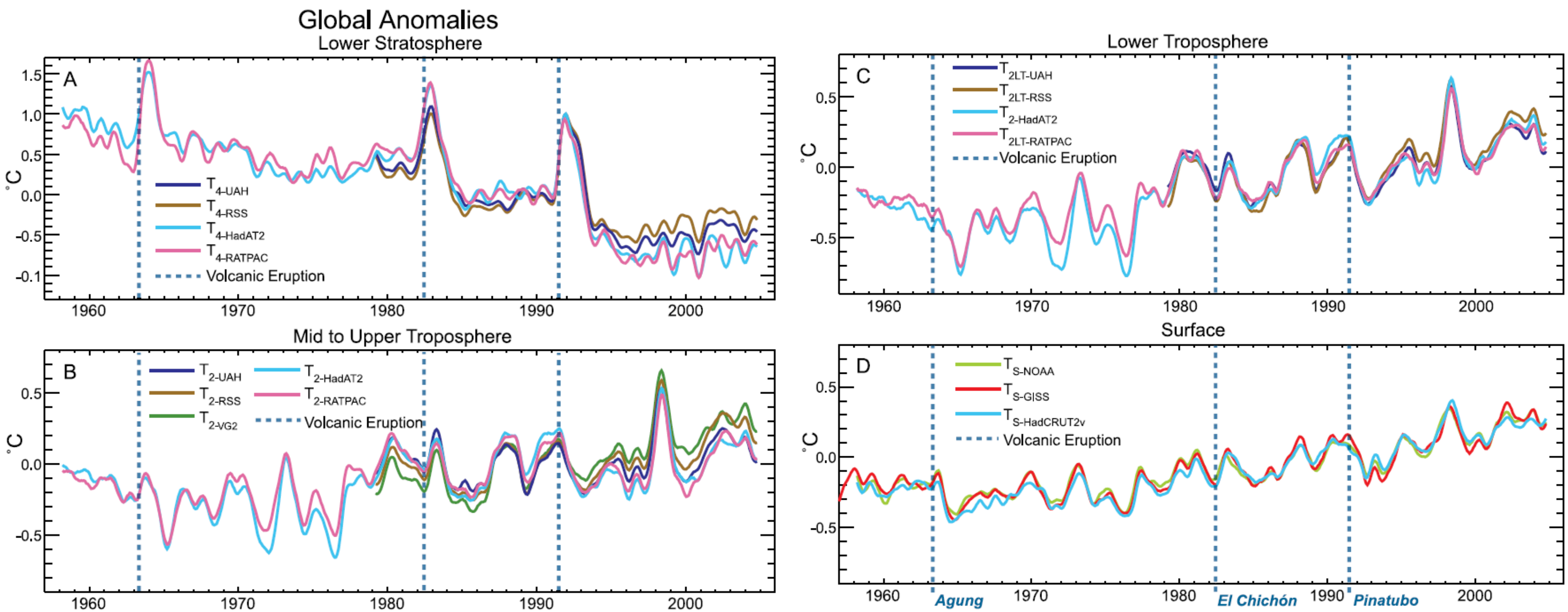


Figure 3.17. *Observed surface and upper-air temperature anomalies (°C). (A) Lower stratospheric T4, (B) Tropospheric T2, (C) Lower tropospheric $T2_{LT}$, from UAH, RSS and VG2 MSU satellite analyses and UKMO HadAT2 and NOAA RATPAC radiosonde observations; and (D) Surface records from NOAA, NASA/GISS and UKMO/CRU (HadCRUT2v). All time series are monthly mean anomalies relative to the period 1979 to 1997 smoothed with a seven-month running mean filter. Major volcanic eruptions are indicated by vertical blue dashed lines. Adapted from Karl et al. (2006).*