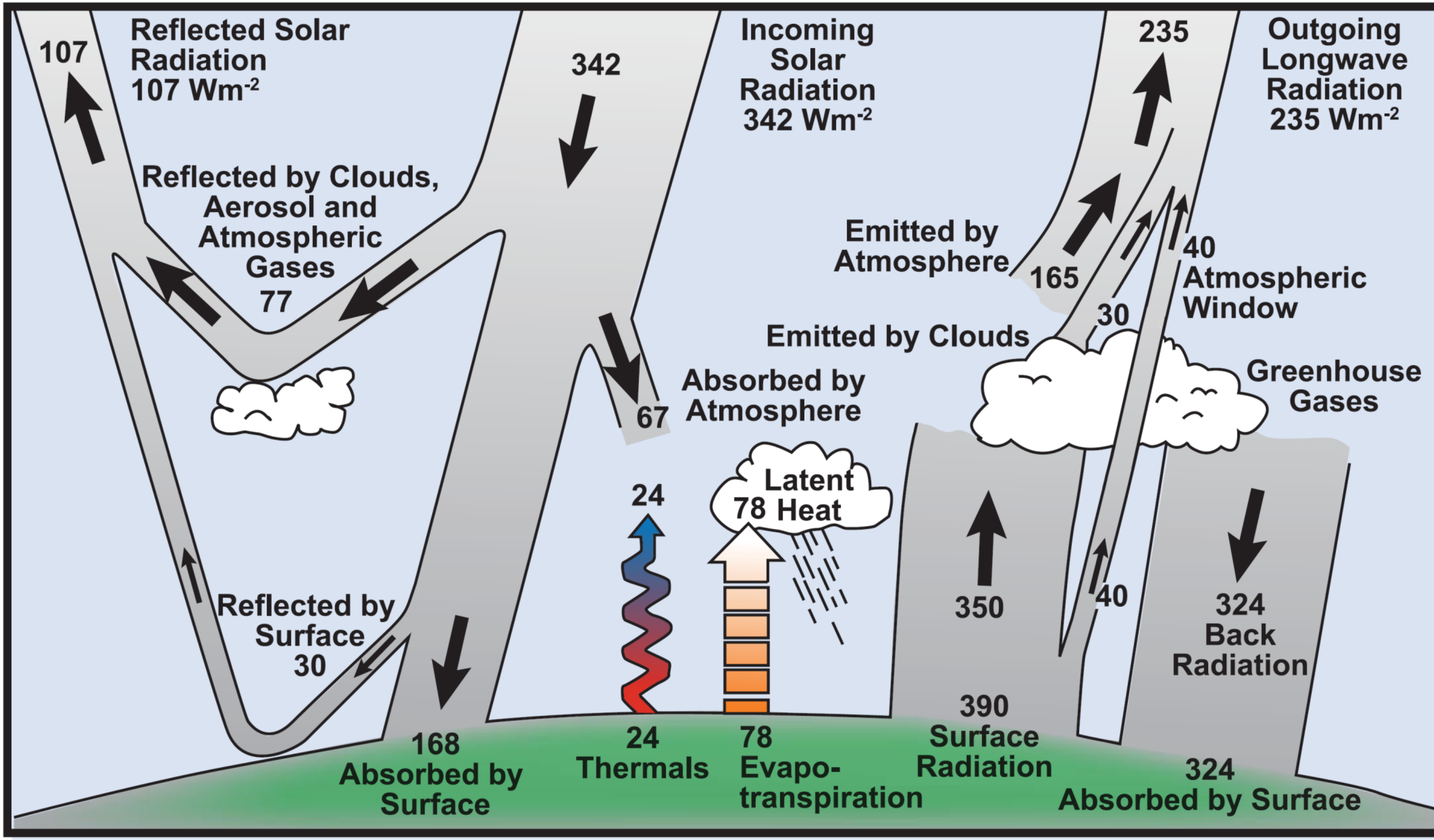
Simple Mathematical Models of the Greenhouse Effect, and Global Warming



FAQ 1.1, Figure 1. *Estimate of the Earth's annual and global mean energy balance. Over the long term, the amount of incoming solar radiation absorbed by the Earth and atmosphere is balanced by the Earth and atmosphere releasing the same amount of outgoing longwave radiation. About half of the incoming solar radiation is absorbed by the Earth's surface. This energy is transferred to the atmosphere by warming the air in contact with the surface (thermals), by evapotranspiration and by longwave radiation that is absorbed by clouds and greenhouse gases. The atmosphere in turn radiates longwave energy back to Earth as well as out to space. Source: Kiehl and Trenberth (1997).*

Mathematical Models

- Scientists often use mathematical and computer models to understand complex systems (like Earth’s climate)
- A model uses equations to represent essential aspects of the system
- The equations describe how one part of the system is quantitatively related to another

Mathematical Models

- Models are necessarily oversimplified and cannot accurately and comprehensively represent all details of a complex system
- Because models are simple, it is easier to see how the real system approximately works
- More equations can be added to a model until it becomes too difficult to understand or simulate on a computer

The Simplest Model

- The Earth absorbs solar radiation
  $S_0 / 4$

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- The Earth has a certain planetary albedo for solar wavelengths

  $(1 - \alpha_p) S_0 / 4$

The Simplest Model

- The Earth absorbs solar radiation

  $S_0 / 4$

- The Earth has a certain planetary albedo for solar wavelengths

  $(1 - \alpha_p) S_0 / 4$

- The Earth emits like a blackbody at IR wavelengths

  σT_e^4

The Simplest Model

- The Earth has no atmosphere

  T_e is the surface temperature

The Simplest Model

- The Earth has no atmosphere

  T_e *is the surface temperature*
- The Earth is in radiative balance

  $(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_e^4$

The Simplest Model

We now have our simple model of Earth's climate:

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_e^4$$

What is the value of T_e?

Rearrange the equation so that T_e is on the left side and everything else is on the right side

What is the value of T_e?

Rearrange the equation so that T_e is on the left side and everything else is on the right side

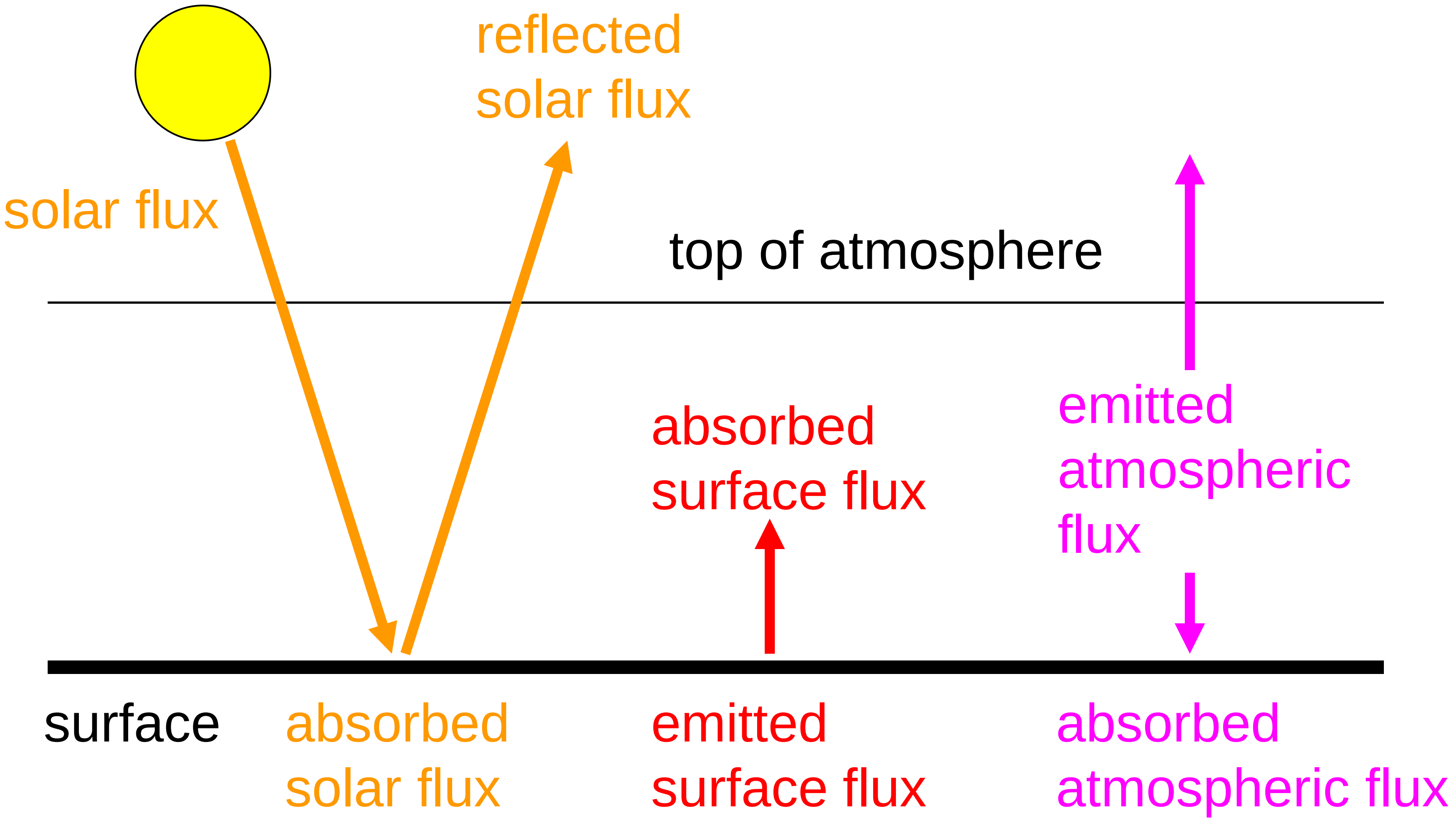
$$T_e = \left[\frac{(1 - \alpha_p) S_0}{4\sigma} \right]^{1/4}$$

Excel spreadsheet

A Simple Atmosphere

- Now add an atmosphere to the model
- This simple atmosphere is perfectly transmissive at solar wavelengths and perfectly absorptive at IR wavelengths
- The atmosphere emits as a perfect blackbody at IR wavelengths
- The atmosphere and surface can have different temperatures T_a and T_s

A Simple Atmosphere



Top Radiative Balance

- No IR radiation emitted by the surface is transmitted to space because all is absorbed by the atmosphere
- Downward radiation flux is $S_0 / 4$
- One component of upward flux is reflected solar radiation: $\alpha_p S_0 / 4$
- The other component of upward flux is IR radiation emitted by the atmosphere: σT_a^4

Top Radiative Balance

For radiative balance:

$$S_0 / 4 = \alpha_p S_0 / 4 + \sigma T_a^4$$

This can be rearranged as:

$$(1 - \alpha_p) S_0 / 4 = \sigma T_a^4$$

Note that T_a is the same as T_e

Atmosphere Radiative Balance

- The atmosphere absorbs no solar radiation
- The atmosphere absorbs all IR radiation emitted by the surface: σT_s^4
- The atmosphere emits IR radiation in both the upward and downward directions: $\sigma T_a^4 + \sigma T_a^4$

Atmosphere Radiative Balance

For radiative balance:

$$\sigma T_s^4 = \sigma T_a^4 \text{ (up)} + \sigma T_a^4 \text{ (down)}$$

This can be rearranged as:

$$\sigma T_s^4 = 2 \sigma T_a^4$$

Surface Radiative Balance

- The surface absorbs some solar radiation: $(1 - \alpha_p)\ S_0 / 4$
- The surface absorbs all IR radiation that the atmosphere emits downward: $\sigma\ T_a^4$
- The surface emits radiation: $\sigma\ T_s^4$

Surface Radiative Balance

For radiative balance:

$$(1 - \alpha_p)\, S_0 / 4 + \sigma\, T_a^4 = \sigma\, T_s^4$$

This can be rearranged as:

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_s^4 - \sigma\, T_a^4$$

The Simple Atmosphere

We now have a simple model of Earth's climate that includes an atmosphere:

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_a^4$$

$$\sigma\, T_s^4 = 2\, \sigma\, T_a^4$$

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_s^4 - \sigma\, T_a^4$$

What are values of T_a and T_s?

Obtain one equation such that T_a is on the left side, T_s is eliminated, and everything else is on the right side

Obtain a second equation such that T_s is on the left side, S_0 and α_p are eliminated, and everything else is on the right side

What are values of T_a and T_s?

$$T_a = \left[\frac{(1 - \alpha_p) S_0}{4\sigma} \right]^{1/4}$$

$$T_s = (2)^{1/4} T_a$$

What are values of T_a and T_s?

- Note that T_s is greater than T_a and T_e
- The atmosphere keeps the surface warmer than it would be if no atmosphere were present
- This is a mathematical model of the greenhouse effect

Excel spreadsheet

A Simple IR Window

- Now allow the atmosphere to transmit to space some of the IR radiation emitted by the surface
- Since the atmosphere is no longer perfectly absorptive at IR wavelengths, it also no longer emits as a perfect blackbody at IR wavelengths

A Simple IR Window

- Let ε be the fraction of IR radiation absorbed by the atmosphere

  surface radiation absorbed by atmosphere

  $\varepsilon \sigma T_s^4$

- Basic physics requires that the ε also be the fraction of IR radiation emitted by the atmosphere relative to a perfect blackbody

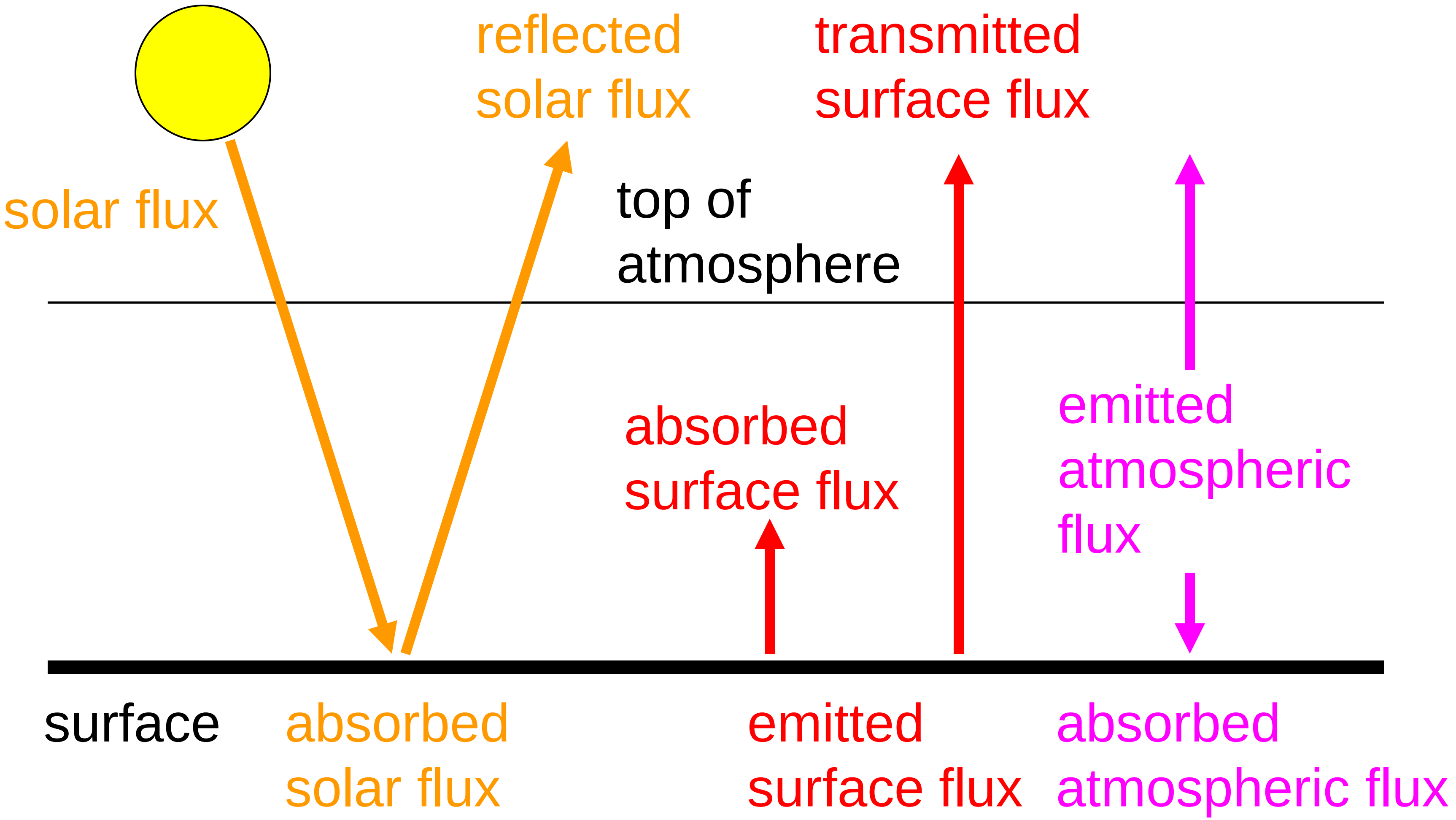
  radiation emitted upward and downward

  $\varepsilon \sigma T_a^4 + \varepsilon \sigma T_a^4$

Emissivity

- The parameter ε is called the emissivity of the atmosphere (the fraction of radiation emitted relative to a blackbody)
- The absorptivity of the atmosphere also has the value of ε (the fraction of incident radiation that is absorbed)

A Simple IR Window



Top Radiative Balance

- Downward radiation flux is S_0 / 4
- One component of upward flux is reflected solar radiation: $\alpha_p S_0 / 4$
- Another component is IR radiation emitted by the atmosphere: $\varepsilon \sigma T_a^4$
- A third component is IR radiation emitted by the surface and not absorbed by the atmosphere: $(1 - \varepsilon) \sigma T_s^4$

Top Radiative Balance

For radiative balance:

$$S_0 / 4 = \alpha_p S_0 / 4 + \varepsilon \sigma T_a^4 + (1 - \varepsilon) \sigma T_s^4$$

This can be rearranged as:

$$(1 - \alpha_p) S_0 / 4 = \varepsilon \sigma T_a^4 + (1 - \varepsilon) \sigma T_s^4$$

Note that T_a is no longer the same as T_e

Atmosphere Radiative Balance

- The atmosphere absorbs no solar radiation
- The atmosphere absorbs some IR radiation emitted by the surface: $\varepsilon \sigma T_s^4$
- The atmosphere emits IR radiation in both the upward and downward directions: $\varepsilon \sigma T_a^4 + \varepsilon \sigma T_a^4$

Atmosphere Radiative Balance

For radiative balance:

$\varepsilon \sigma T_s^4 = \varepsilon \sigma T_a^4$ (up) $+ \varepsilon \sigma T_a^4$ (down)

This can be rearranged as:

$\varepsilon \sigma T_s^4 = 2 \varepsilon \sigma T_a^4$

Surface Radiative Balance

- The surface absorbs some solar radiation: $(1 - \alpha_p)\ S_0 / 4$
- The surface absorbs all IR radiation that the atmosphere emits downward: $\varepsilon\ \sigma\ T_a^4$
- The surface emits radiation: $\sigma\ T_s^4$

Surface Radiative Balance

For radiative balance:

$$(1 - \alpha_p)\, S_0 / 4 + \varepsilon\, \sigma\, T_a^4 = \sigma\, T_s^4$$

This can be rearranged as:

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_s^4 - \varepsilon\, \sigma\, T_a^4$$

The Simple IR Window

We now have a simple model of Earth's climate that includes an atmosphere that is partially transmissive at IR wavelengths:

$$(1 - \alpha_p)\, S_0 / 4 = \varepsilon\, \sigma\, T_a^4 + (1 - \varepsilon)\, \sigma\, T_s^4$$

$$\varepsilon\, \sigma\, T_s^4 = 2\, \varepsilon\, \sigma\, T_a^4$$

$$(1 - \alpha_p)\, S_0 / 4 = \sigma\, T_s^4 - \varepsilon\, \sigma\, T_a^4$$

How are T_s and ε related?

Obtain one equation such that T_s is on the left side, T_a is eliminated, and everything else is on the right side

How are T_s and ε related?

Obtain one equation such that T_s is on the left side, T_a is eliminated, and everything else is on the right side

$$T_s = \left[\frac{(1 - \alpha_p) S_0}{(4 - 2\varepsilon)\sigma} \right]^{1/4}$$

How are T_s and ε related?

- Note that $\varepsilon = 0$ corresponds to no effective atmosphere and $\varepsilon = 1$ corresponds to a perfectly absorbing atmosphere
- For $0 < \varepsilon < 1$, T_s is warmer than T_e and colder than T_s for the perfectly absorbing atmosphere
- Changes in ε represent changes in greenhouse gas concentrations

How are T_s and ε related?

- What value of ε will produce a value of T_s equal to current global surface temperature (about 288 K or 15°C)?
- How close is this to the real fraction of radiation absorbed by the atmosphere?
- How much does ε need to change to produce a 1 K increase in T_s?
- What about 2 K? 5 K? A decrease in T_s?

Excel spreadsheet

Non-Equilibrium

- The climate does not have an instantaneous response to a change in emissivity
- What is the transient behavior of the atmosphere before it comes to equilibrium?

A Simple Climate Model

- Let the Earth be covered by a “swamp” ocean with uniform depth *h*, density ρ, and specific heat *c*

  thermal inertia of ocean = $\rho\ c\ h$

- Let *F* be the net radiation flux at the Earth’s surface

  $F = (1 - \alpha_p)\ S_0 / 4 + \varepsilon\ \sigma\ T_a^4 - \sigma\ T_s^4$

A Simple Climate Model

Let ΔT_s be the change in temperature across time interval Δt

$$\Delta T_s = F \, \Delta t \, / \, (\rho \, c \, h)$$

where

$$F = (1 - \alpha_p) \, S_0 \, / \, 4 + \varepsilon \, \sigma \, T_s^4 \, / \, 2 - \sigma \, T_s^4$$

A Simple Climate Model

- Assume ε is known as a function of time
- If T_s is known at time t_0, it is simple to calculate T_s at time $t_0 + \Delta t$
- The value of T_s at time $t_0 + \Delta t$ can then be used to calculate T_s at time $t_0 + 2\Delta t$
- Etc.

A Simple Climate Scenario

- Let the climate initially be in equilibrium
- Let ε change instantaneously from 0.8 to 0.85 and thereafter remain constant
- Let *h* be 4000 m (approximate average ocean depth)
- Let ρ be 1025 kg m^{-3} and *c* be 3850 J kg^{-1} K^{-1} (typical values for seawater)

Excel spreadsheet